A reduced basis method for accelerating parameterized non-linear microstructures

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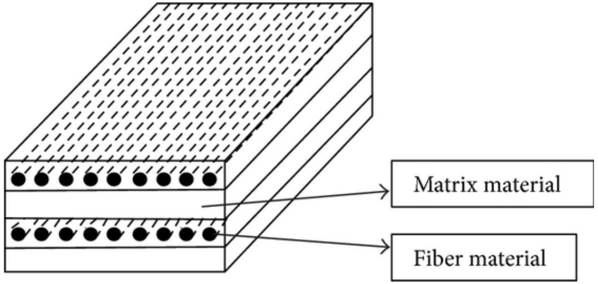
Motivation

- Understanding structure-property relations
  - What is the influence of the microstructure onto macroscopic behavior?
  - What is the optimal microstructure for a given application?

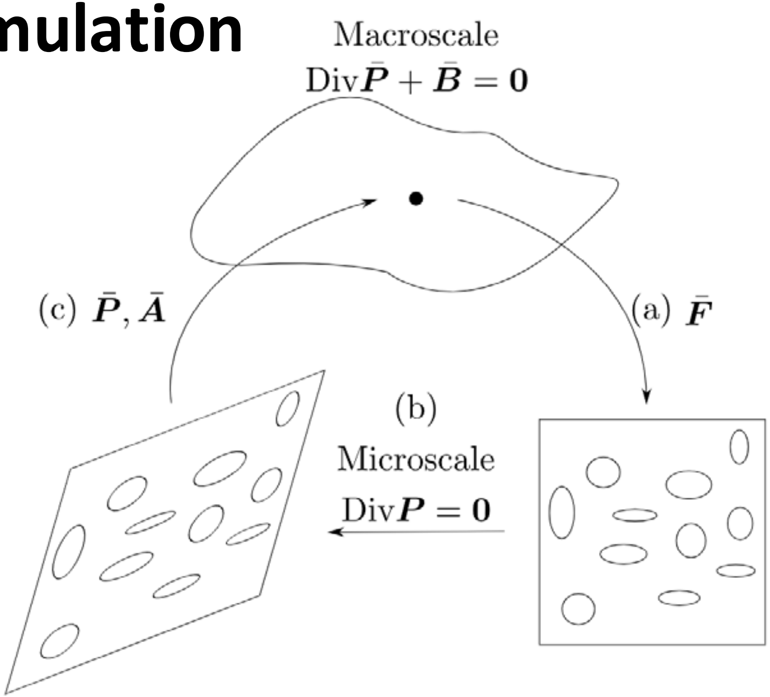
- Applications
  - **Design**: Composite Materials, Metamaterials
  - **Reliability Analysis**: Effect of microscopic variations
  - **Design of Production Processes**: Effect of external factors



(https://www.researchgate.net/figure/Composite-material-with-fiber-and-matrix_fig13_258393869)

Two-Scale Simulation

$$\boxed{\bar{\boldsymbol{P}} = \langle \boldsymbol{P} \rangle = |\Omega|^{-1} \int_{\Omega} \boldsymbol{P} dV}$$



Full two-scale simulation → **too expensive** (multi-query contexts such as optimization, material design, etc.)

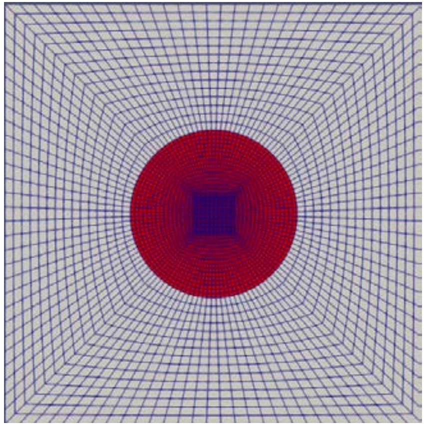
[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Microscopic Problem

$$\mathrm{Div}\,\boldsymbol{P}(\boldsymbol{u}) = \mathbf{0} \quad \text{on } \Omega$$

$$\boldsymbol{u} = \underbrace{(\bar{\mathrm{U}} - \mathrm{I})\boldsymbol{X}}_{\text{mean}} + \underbrace{\boldsymbol{w}(\boldsymbol{X})}_{\text{fluctuation}}$$

$$\boldsymbol{w} \text{ periodic on } \partial\Omega$$



Several problem parameters:

- **3 loading parameters**
- Each phase has 2 **material parameters**
- Each phase can **be geometrically parameterized**
- (Interfaces between phases could be modelled)

Each phase modelled as hyperelastic material

$$W(\boldsymbol{F}, \boldsymbol{\lambda}) = C_1(\mathrm{Tr}(\boldsymbol{F}^T\boldsymbol{F}) - 3 - 2\ln(\det \boldsymbol{F})) + D_1(\det \boldsymbol{F} - 1)^2$$

$$\boldsymbol{P}(\boldsymbol{F}, \boldsymbol{\lambda}) = \frac{\partial W}{\partial \boldsymbol{F}} \qquad \boldsymbol{\lambda} = [C_1, D_1]^T$$

$$\boldsymbol{A}(\boldsymbol{F}, \boldsymbol{\lambda}) = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}}$$

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Methodology

Related Works

- Fast Fourier Transform
    - [Moulinec, H., Suquet, P., 1997, CMAME]

- (Nonuniform) Transformation Field Analysis
    - [Michel, J., Suquet, P., 2003, Int J Solids Struct]
    - [Fritzen, F., Leuschner, M., 2013, CMAME]

- Reduced Basis Method
    - [Hernández, J.A., et al., 2014, CMAME]
    - [Radermacher, A., Reese, S., 2016, Int J Nume Meth Eng]
    - [Yvonnet, J., He, Q., 2007, JCP]

Only considered loading parameters:
- Can speed up one forward simulation
- Cannot be used for optimization

Methodology

- Learn effective constitutive model from precomputed microscopic simulations
  → reduce to two **single-scale** problems

  (Offline Stage)

- Learn regression model for **effective stress and effective stiffness**

$$\bar{P}(\bar{\mathrm{U}}, \boldsymbol{\lambda}, \boldsymbol{\mu}), \quad \bar{A} = \frac{\partial \bar{P}(\bar{\mathrm{U}}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial \bar{\mathrm{U}}}$$

$\bar{\mathrm{U}}$: Loading parameters
$\boldsymbol{\lambda}$: Material parameters
$\boldsymbol{\mu}$: Geometrical parameters

- **Effective sensitivities**

$$\frac{\partial \bar{P}(\bar{\mathrm{U}}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial \boldsymbol{\lambda}}, \frac{\partial \bar{P}(\bar{\mathrm{U}}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial \boldsymbol{\mu}}$$

[Wu, L., et al., 2020, CMAME 369]
[Mozaffar, M., et al., 2019, PNAS 116]

Methodology

- Many samples need to be generated
- Microscopic quantities cannot be recovered
- Physics might not be fulfilled

- Instead: Approximation of the microscopic **stress field**

$$\boxed{\boldsymbol{P}(\boldsymbol{X};\bar{\mathrm{U}},\boldsymbol{\lambda},\boldsymbol{\mu})}$$

$$\bar{\boldsymbol{P}}(\bar{\mathrm{U}},\boldsymbol{\lambda},\boldsymbol{\mu}), \quad \bar{\boldsymbol{A}} = \frac{\partial \bar{\boldsymbol{P}}(\bar{\mathrm{U}},\boldsymbol{\lambda},\boldsymbol{\mu})}{\partial \bar{\mathrm{U}}}$$

$$\frac{\partial \bar{\boldsymbol{P}}(\bar{\mathrm{U}},\boldsymbol{\lambda},\boldsymbol{\mu})}{\partial \boldsymbol{\lambda}}, \frac{\partial \bar{\boldsymbol{P}}(\bar{\mathrm{U}},\boldsymbol{\lambda},\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}$$

$\bar{\mathrm{U}}$: Loading parameters
$\boldsymbol{\lambda}$: Material parameters
$\boldsymbol{\mu}$: Geometrical parameters

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Methodology

Idea: Reduced Basis Method

- Approximate the microscopic stress field with a linear combination of global basis functions with parameter-dependent coefficients

$$\boldsymbol{P} \approx \boldsymbol{P}^{\mathrm{RB}}(\boldsymbol{X}; \bar{\mathrm{U}}, \boldsymbol{\lambda}) = \sum_{l=1}^{L} \alpha_l(\bar{\mathrm{U}}, \boldsymbol{\lambda}) \boldsymbol{B}_l(\boldsymbol{X})$$

- Construct reduced basis functions $\boldsymbol{B}_l(\boldsymbol{X})$ from a collection of precomputed stress fields with **proper orthogonal decomposition** (POD)

Methodology

$$\int_\Omega \boldsymbol{B}_l(\boldsymbol{X})dV$$ can be computed offline!

Effective Quantities

$$\bar{\boldsymbol{P}} = \langle \boldsymbol{P}^{\mathrm{RB}} \rangle = |\Omega|^{-1} \int_\Omega \boldsymbol{P}^{\mathrm{RB}} dV = |\Omega|^{-1} \sum_{l=1}^{L} \boxed{\alpha_l(\bar{\mathbf{U}}, \boldsymbol{\lambda})} \int_\Omega \boldsymbol{B}_l(\boldsymbol{X})dV$$

Stiffness/Sensitivities depend on derivatives of $\alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\lambda})$

$\Rightarrow$ Construct $\hat{\alpha}_l(\bar{\mathbf{U}}, \boldsymbol{\lambda}), \dfrac{\partial \hat{\alpha}_l(\bar{\mathbf{U}}, \boldsymbol{\lambda})}{\partial \bar{\mathbf{U}}}, \dfrac{\partial \hat{\alpha}_l(\bar{\mathbf{U}}, \boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}}$ with Gaussian Process Regression

Automatic relevance determination (ARD) squared exponential kernel

$$k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}') = \sigma_f^2 \exp\left(-\frac{1}{2} \sum_{k=1}^{d} \frac{(X_k - X_k')^2}{l_k^2} \right)$$

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]
[Guo, M., Hesthaven, J., 2018. CMAME 341]
[Kast, M., Guo, M., Hesthaven, J., 2020. CMAME 364]

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

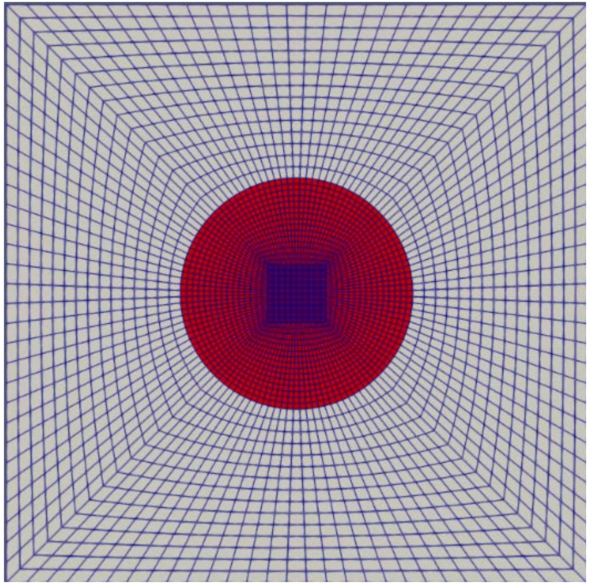
Fiber reinforced microstructure

- 4321 eight-node elements with 4 quadrature points
- Volume fraction of fiber is 12.56%
- Matrix material $C_1^{\mathrm{mat}} = D_1^{\mathrm{mat}} = 1$
- Fiber material $C_1^{\mathrm{fib}} = D_1^{\mathrm{fib}} \in [50, 150]$
- Deformations $\bar{\mathbf{U}} - \mathbf{I} \in [-0.3, 0.3]$
- 200 training snapshots (Sobol sequence)
- 500 testing snapshots (uniformly random)
- P-PODGPR Error

$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\|\bar{\mathbf{P}}^{\mathrm{HF}} - \hat{\bar{\mathbf{P}}}^{\mathrm{RB}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{P}}^{\mathrm{HF}}\right\|_{\mathrm{F}}}$$

- Projection Error

$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\|\bar{\mathbf{P}}^{\mathrm{HF}} - \bar{\mathbf{P}}^{\mathrm{RB}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{P}}^{\mathrm{HF}}\right\|_{\mathrm{F}}}$$



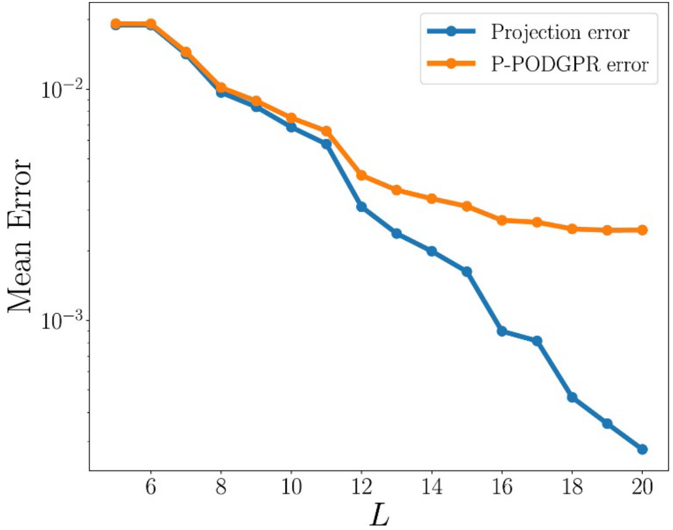
$$W(\boldsymbol{F}, \boldsymbol{\lambda}) = C_1(\mathrm{Tr}(\boldsymbol{F}^T\boldsymbol{F}) - 3 - 2\ln(\det \boldsymbol{F}))$$
$$+ D_1(\det \boldsymbol{F} - 1)^2$$

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

Test error of 500 testing snapshots trained with 200 training snapshots



Mean error ≈ 0.003

P-PODGPR Error

$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\|\bar{\mathbf{P}}^{\mathrm{HF}} - \hat{\bar{\mathbf{P}}}^{\mathrm{RB}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{P}}^{\mathrm{HF}}\right\|_{\mathrm{F}}}$$

Projection Error

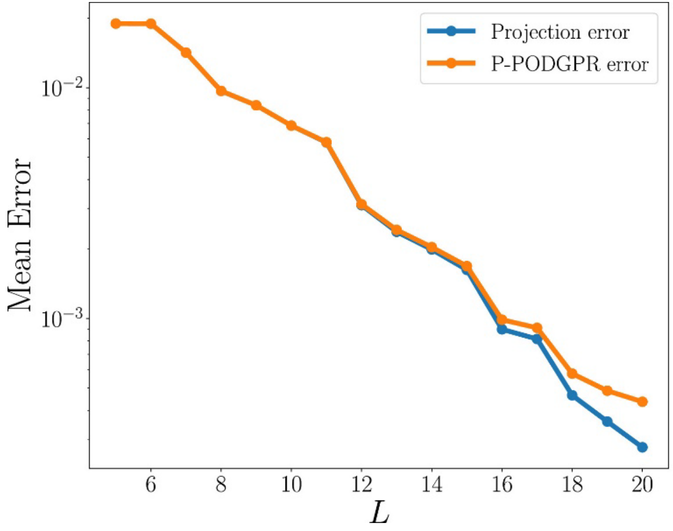
$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\|\bar{\mathbf{P}}^{\mathrm{HF}} - \bar{\mathbf{P}}^{\mathrm{RB}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{P}}^{\mathrm{HF}}\right\|_{\mathrm{F}}}$$

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

Test error of 500 testing snapshots trained with 500 training snapshots



P-PODGPR Error

$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\|\bar{\mathbf{P}}^{\mathrm{HF}} - \hat{\bar{\mathbf{P}}}^{\mathrm{RB}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{P}}^{\mathrm{HF}}\right\|_{\mathrm{F}}}$$

Projection Error

$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\|\bar{\mathbf{P}}^{\mathrm{HF}} - \bar{\mathbf{P}}^{\mathrm{RB}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{P}}^{\mathrm{HF}}\right\|_{\mathrm{F}}}$$

Mean error ≈ 0.0004

Numerical Examples

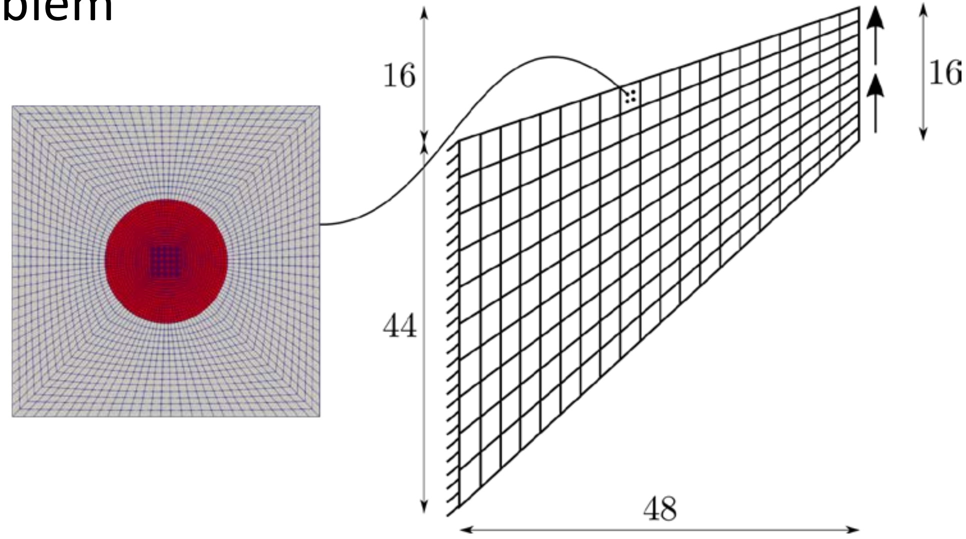
[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

- 200 four-node elements with 4 quadrature points
- Vertical traction of 0.1 on right edge
- Left edge is fixed
- Matrix material $C_1^{\mathrm{mat}} = D_1^{\mathrm{mat}} = 1$
- Fiber material $C_1^{\mathrm{fib}} = D_1^{\mathrm{fib}} = 100$
- Surrogate model trained with 200 snapshots
- Relative error to compare stresses

$$\epsilon_{P_{yx}} := |P_{yx}^{\mathrm{FE2}} - P_{yx}^{\mathrm{ROM}}| / \langle |P_{yx}^{\mathrm{FE2}}| \rangle$$

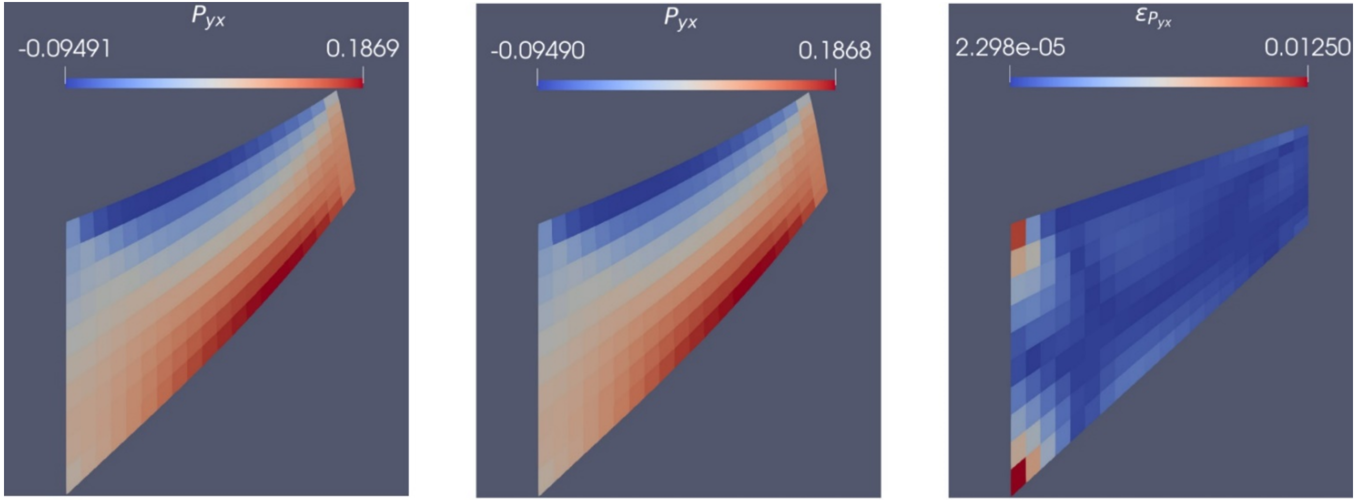
$\langle |P_{yx}^{\mathrm{FE2}}| \rangle$: Averaged absolute stress



$$W(\boldsymbol{F}, \boldsymbol{\lambda}) = C_1(\mathrm{Tr}(\boldsymbol{F}^T\boldsymbol{F}) - 3 - 2\ln(\det \boldsymbol{F}))$$
$$+ D_1(\det \boldsymbol{F} - 1)^2$$

Numerical Examples

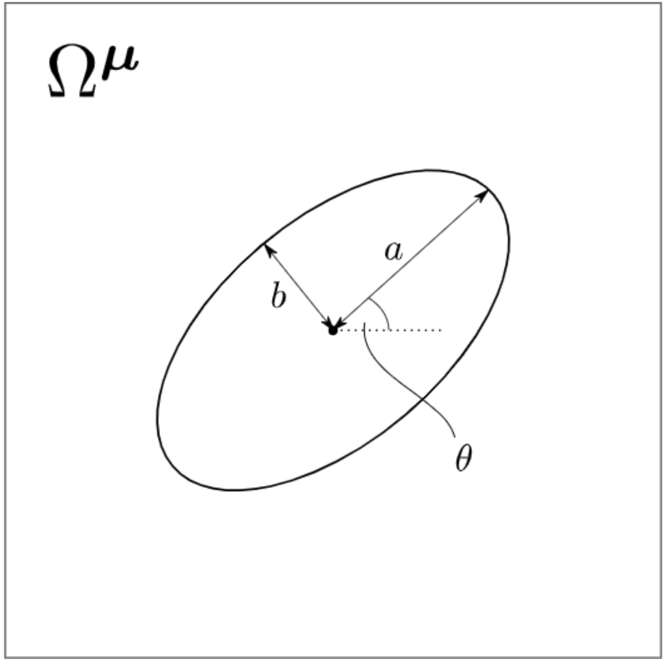
[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]



(a) FE^2 (b) FE with P-PODGPR (c) Relative error

	FE^2	FE with P-PODGPR
Snapshot generation	-	60 min
POD + GPR	-	1 min
Online simulation	4800 min	1 min

Geometrical Parameters



a : Semi-major axis
b : Semi-minor axis
θ : Rotation angle

Geometrical Parameters

Problems:

- Different geometry generally leads to different meshes used → solution for different geometries known at different discrete points → needs interpolation
- Poor performance of POD

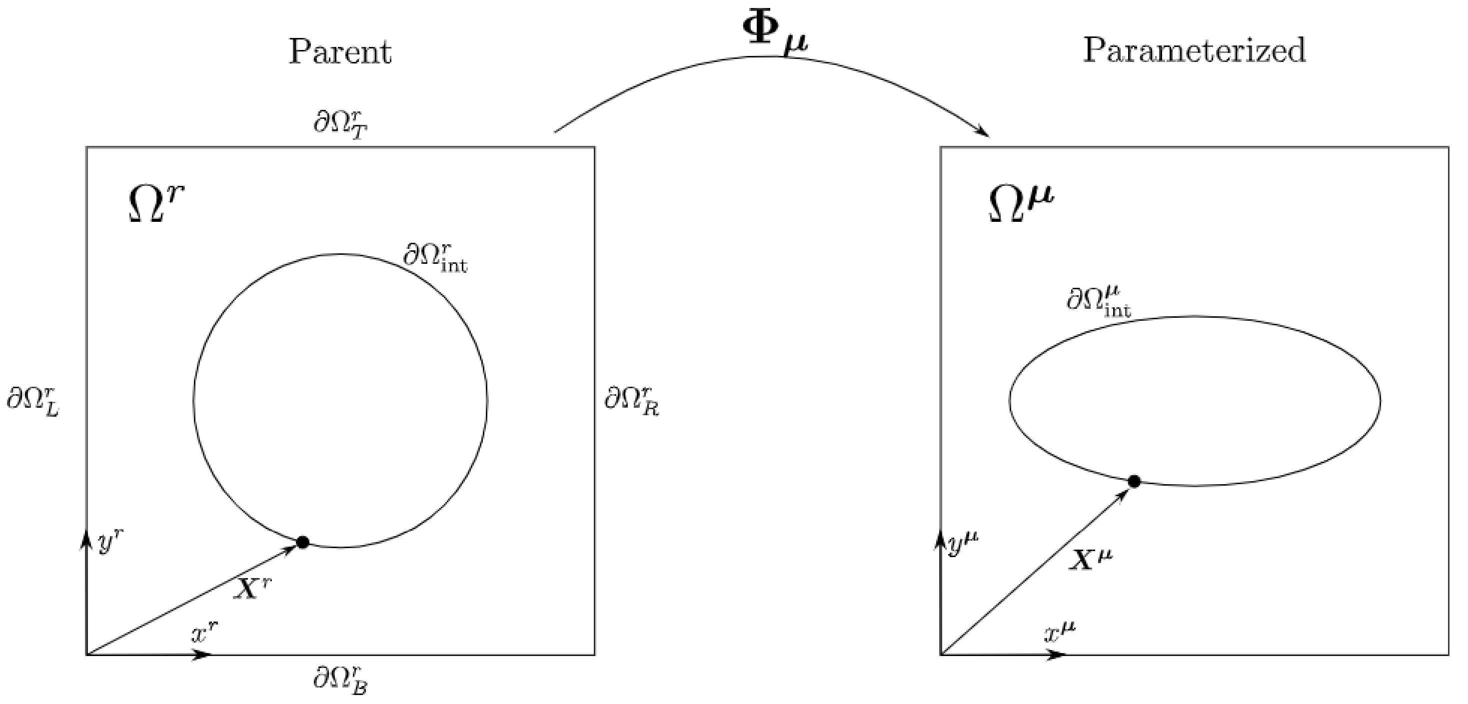
Approach:

- Transformation of snapshots from a parent domain

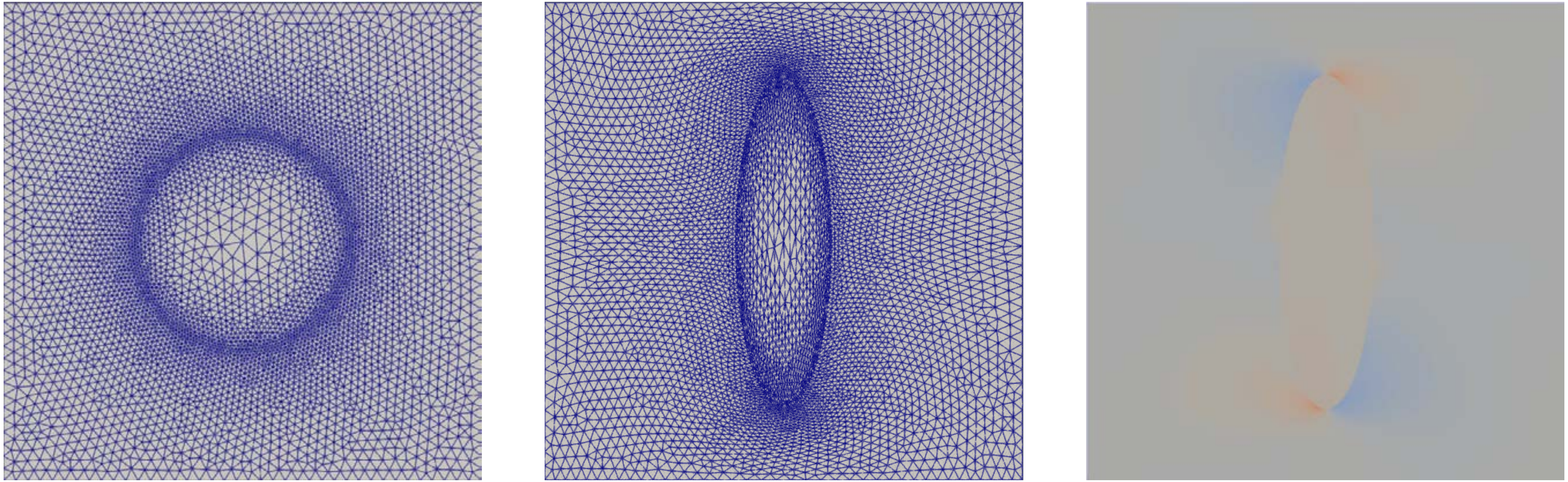
$$\boldsymbol{\Phi}_{\boldsymbol{\mu}} : \Omega^r \to \Omega^{\boldsymbol{\mu}}, \boldsymbol{X}^r \mapsto \boldsymbol{X}^{\boldsymbol{\mu}}$$

$$\boldsymbol{P}^r(\boldsymbol{X}^r) := \boldsymbol{P}(\boldsymbol{\Phi}_{\boldsymbol{\mu}}(\boldsymbol{X}^r))$$

Geometrical Parameters



Geometrical Parameters



Fixing loading parameters and semi-major and minor axis, varying rotation angle

Summary

- Approximation of stress field based on proper orthogonal decomposition and Gaussian process regression
- Validated on 2D composite microstructure and a two-scale application
- High accuracy and online speedup
- Extension to geometrically parameterized microstructures

Outlook

- Two-scale optimization with surrogate model
- Reduce number of samples needed for training
- More complicated material models (Plasticity, Damage)

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Methodology

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Proper Orthogonal Decomposition (POD)

1. Generate samples $\left[(\bar{\boldsymbol{F}}^{(1)}, \boldsymbol{\lambda}^{(1)}), (\bar{\boldsymbol{F}}^{(2)}, \boldsymbol{\lambda}^{(2)}), \ldots, (\bar{\boldsymbol{F}}^{(N_{\text{pod}})}, \boldsymbol{\lambda}^{(N_{\text{pod}})})\right]$
2. Compute and collect stress snapshots $\mathbf{S} = \left[\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \ldots, \mathbf{P}^{(N_{\text{pod}})}\right]$
3. Compute correlation matrix

$$\mathrm{C}_{ij} = \left(\boldsymbol{P}^{(i)}, \boldsymbol{P}^{(j)}\right)_{L^2(\Omega)} = \int_\Omega \boldsymbol{P}^{(i)} : \boldsymbol{P}^{(j)} dV$$

4. Compute the eigenvalues λ_l and eigenvectors $\mathbf{v}_l$ of $\mathbf{C}$
5. Compute the basis functions

$$\mathbf{B}_l = \frac{1}{\sqrt{\lambda_l}} \mathbf{S}\mathbf{v}_l \qquad (\boldsymbol{B}_i, \boldsymbol{B}_j)_{L^2(\Omega)} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

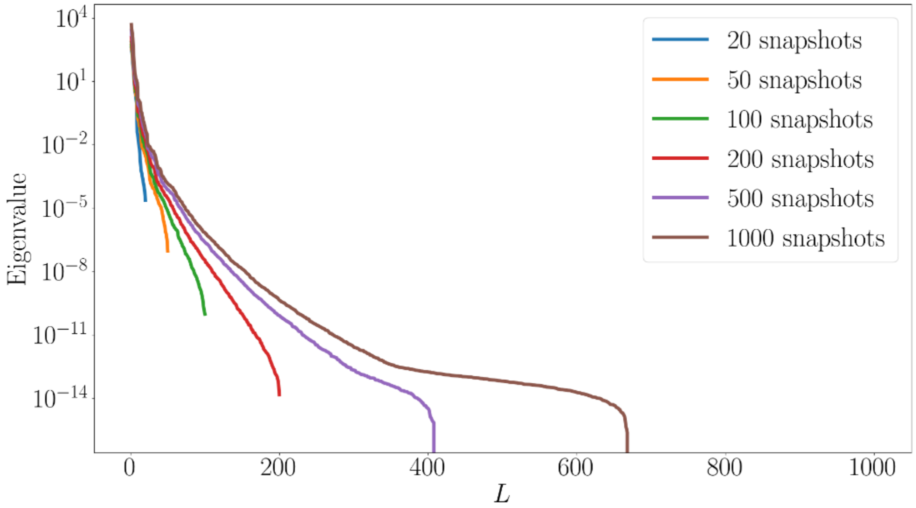
$$\Rightarrow \boldsymbol{P}^{\text{RB}}(\boldsymbol{X}, \bar{\boldsymbol{F}}, \boldsymbol{\lambda}) = \sum_{l=1}^{L} \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\lambda}) \boldsymbol{B}_l(\boldsymbol{X})$$

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

POD for different numbers of training snapshots



With $L = 20,\quad \dfrac{\sum_{i=1}^{L} \lambda_i}{\sum_{i=1}^{N} \lambda_i} = 0.9999$

Methodology

[Guo, M. and Hesthaven, J.S., 2018, CMAME 341]
[Rasmussen, C.E., 2003, *Summer school on machine learning*]

Gaussian Process Regression (GPR)

Given some data $\{\mathbf{X}^{(i)}, y^{(i)}\}_{i=1}^{N}$, we want to find a scalar regression function which is distributed as a Gaussian Process with zero mean function and kernel $k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}')$

$$f \sim \mathcal{GP}(0, k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}'))$$

Automatic relevance determination (ARD) squared exponential kernel

$$k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}') = \sigma_f^2 \exp\left(-\frac{1}{2}\sum_{k=1}^{d}\frac{(X_k - X_k')^2}{l_k^2}\right)$$

Methodology

[Guo, M. and Hesthaven, J.S., 2018, CMAME 341]
[Rasmussen, C.E., 2003, *Summer school on machine learning*]

Gaussian Process Regression (GPR)

Given some data $\{\mathbf{X}^{(i)}, y^{(i)}\}_{i=1}^{N}$, we want to find a scalar regression function which is distributed as a Gaussian Process with zero mean function and kernel $k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}')$

$$
\begin{aligned}
f^* | (\mathbf{X}, \mathbf{y}) &\sim \mathcal{GP}(m^*, k^*), \\
m^*(\mathbf{x}) &= \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x})^T \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X})^{-1} \mathbf{y}(\mathbf{X}), \\
k^*(\mathbf{x}, \mathbf{x}') &= k_{\boldsymbol{\theta}}(\mathbf{x}, \mathbf{x}') - \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x})^T \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X})^{-1} \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x}')
\end{aligned}
$$

where
$$
\begin{aligned}
\mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x}) &= [k_{\boldsymbol{\theta}}(\mathbf{X}^{(1)}, \mathbf{x}), k_{\boldsymbol{\theta}}(\mathbf{X}^{(2)}, \mathbf{x}), \ldots, k_{\boldsymbol{\theta}}(\mathbf{X}^{(N)}, \mathbf{x})]^T \\
\mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}) &= [k_{\boldsymbol{\theta}}(\mathbf{X}^{(i)}, \mathbf{X}^{(j)})]_{i,j=1}^{N} \\
\mathbf{y}(\mathbf{X}) &= [f(\mathbf{X}^{(1)}), f(\mathbf{X}^{(2)}), \ldots, f(\mathbf{X}^{(N)})]^T
\end{aligned}
$$

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

- Comparison with a neural network (Multilayer Perceptron) that is trained with pairs of $\{(\bar{\boldsymbol{F}}, \boldsymbol{\lambda})^{(i)}, \bar{\boldsymbol{P}}^{(i)}\}_{i=1}^{N}$ with $N = 500$ training snapshots
- Trained with Adam optimizer with learning rate of 10^{-4}, batch size of 32 and Mean Squared Error Loss function for different architectures
- ELU activation function was applied after every layer apart from last layer

Architecture	Validation Loss	$\epsilon_{\bar{\mathbf{P}}}^{\text{mean}}$	$\epsilon_{\bar{\mathbf{P}}}^{\text{max}}$
$N_h = 1, N_n = 20$	1.7×10^{-6}	0.0077	0.0368
$N_h = 1, N_n = 50$	8.55×10^{-7}	0.0056	0.0289
$N_h = 2, N_n = 20$	5.36×10^{-7}	0.0047	**0.0176**
$N_h = 2, N_n = 50$	$\mathbf{2.97 \times 10^{-7}}$	**0.0039**	0.0206
$N_h = 2, N_n = 100$	7.91×10^{-7}	0.0052	0.0315

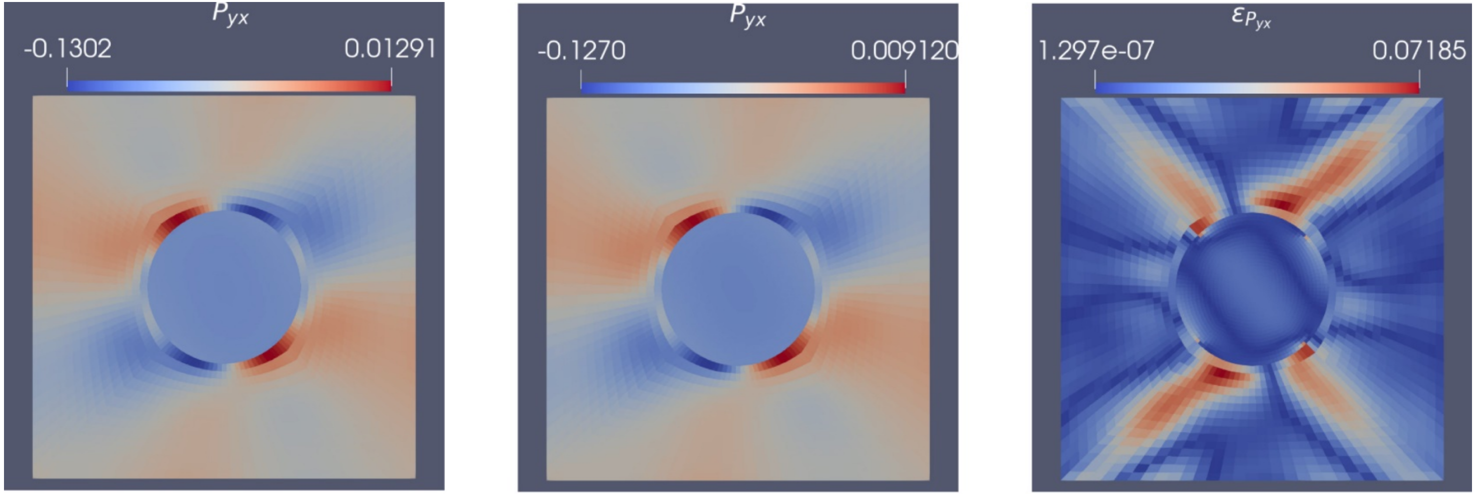
$\Rightarrow$ P-PODGPR outperforms all neural networks

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

Microscopic stress component P_{yx} at point A



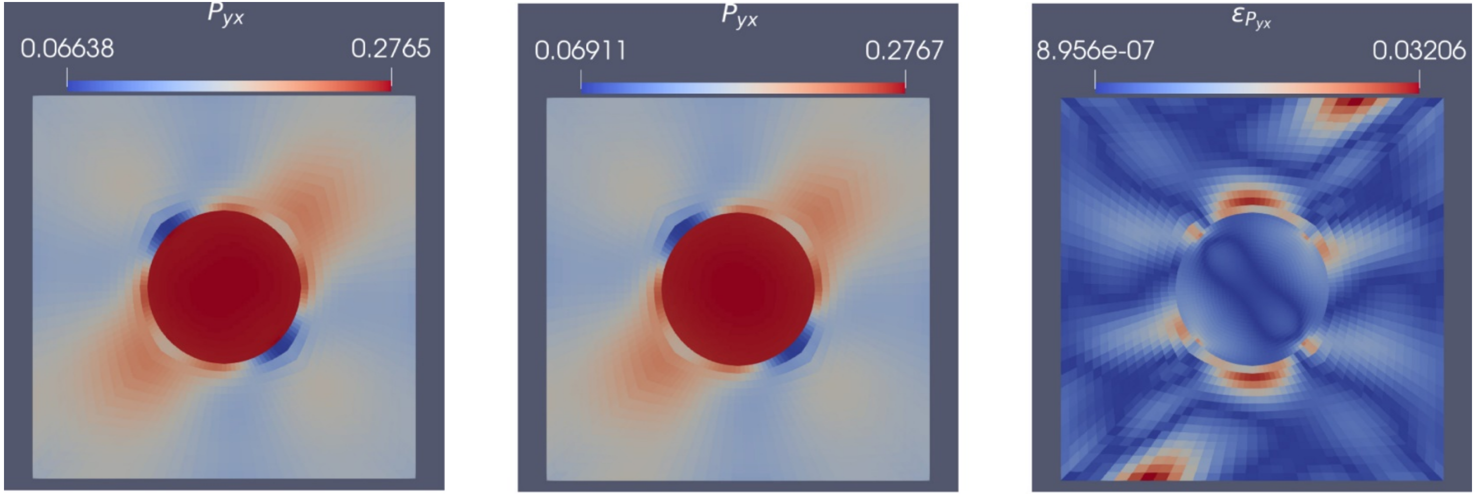
(a) FE^2 (b) FE with P-PODGPR (c) Relative error

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

Microscopic stress component P_{yx} at point B



(a) FE^2 (b) FE with P-PODGPR (c) Relative error

Current Research: Geometrical Parameters

Related Works

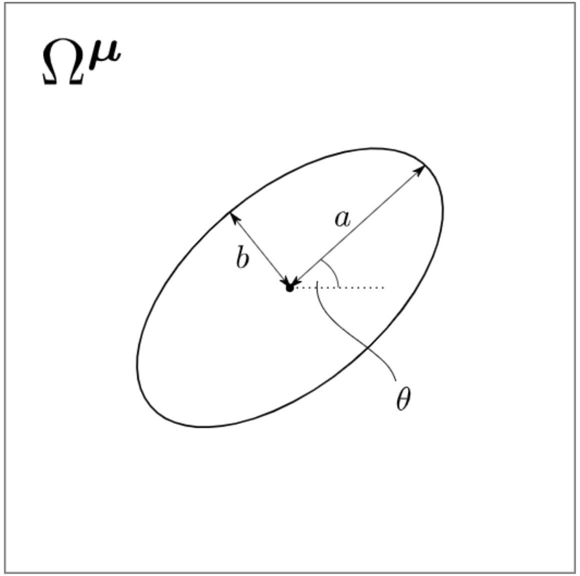
- Analytical Transformation
  - [Moreau, A., et al., 2021, Acta Mechanica]
  - [Rozza, G., Huynh, D.B.P., Patera, A.T., 2008, Arch. Comput. Meth Eng.]

- Shape Deformation Techniques
  - [Sieger, D., Menzel, S., Botsch, M., 2015, Springer]
  - [Rozza, G., et al., 2020, De Gruyter]

- Image Registration Method
  - [Taddei, T., 2020, SIAM SC]

Current Research: Geometrical Parameters

Example: Rotating Elliptical Fiber



Auxiliary Problem

$$\begin{aligned} \mathrm{Div}(\mathbb{C} : \nabla^s \boldsymbol{d}) &= \mathbf{0} && \text{on } \Omega^r, \\ \boldsymbol{d} &= \mathbf{0} && \text{on } \partial\Omega^r, \\ \boldsymbol{d} &= \boldsymbol{X}^{\mu} - \boldsymbol{X}^r && \text{on } \partial\Omega^r_{\text{interface}}. \end{aligned}$$

Interface deformation is known and can be prescribed!

$$\boldsymbol{X}^{\mu} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a/r & 0 \\ 0 & b/r \end{bmatrix} \begin{pmatrix} \tilde{r}\cos(\tilde{\theta}-\theta) \\ \tilde{r}\sin(\tilde{\theta}-\theta) \end{pmatrix} \qquad \begin{aligned} \tilde{r} &= \sqrt{\boldsymbol{X}^{r^T}\boldsymbol{X}^r} \\ \tilde{\theta} &= \mathrm{arctan2}(y^r, x^r) \end{aligned}$$

3 geometrical parameters + 4 loading parameters

TU/e

Current Research: Geometrical Parameters

Problems:

- Different meshes used for different snapshots
- Solutions do not lie in linear subspace → poor performance of POD

Solution:

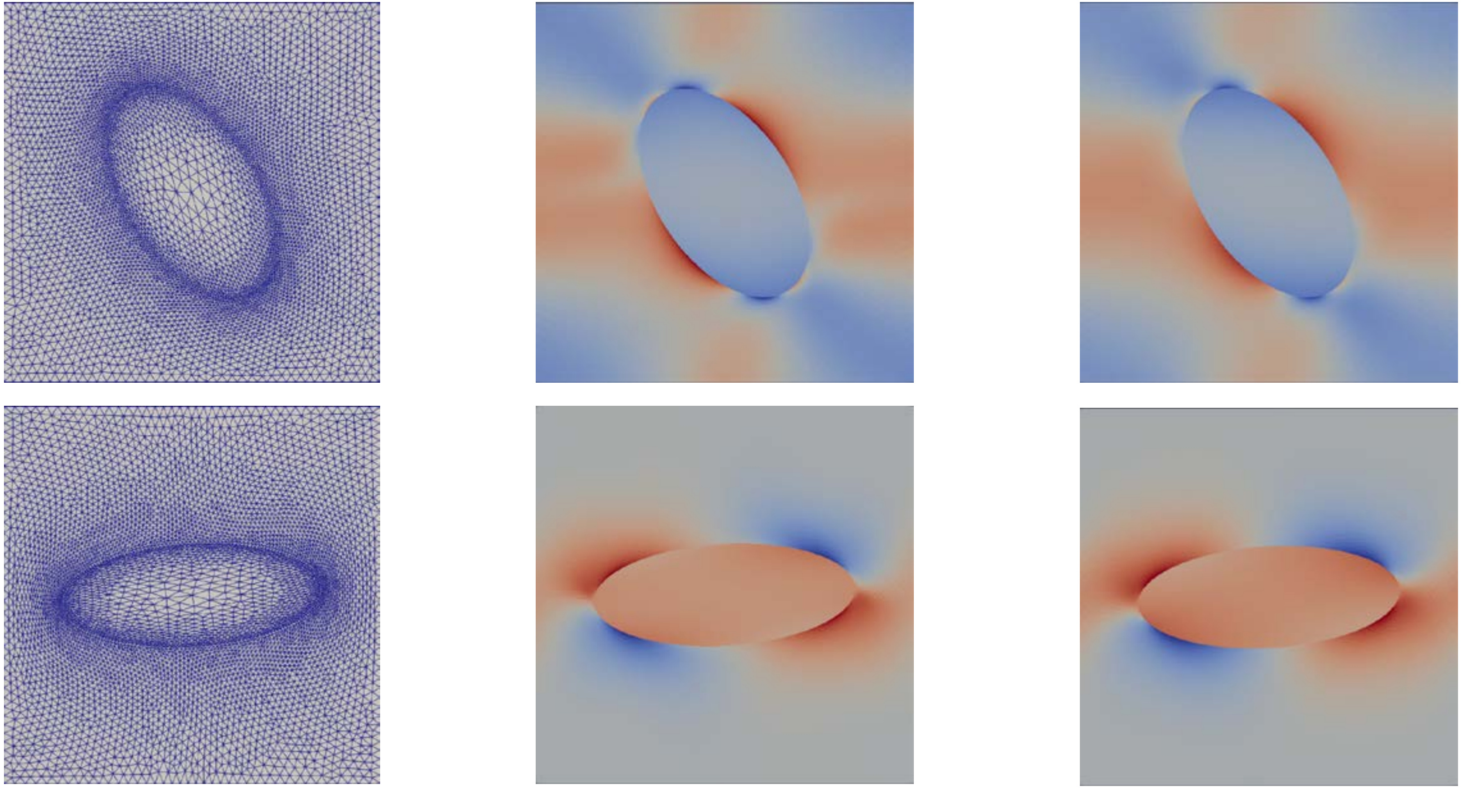
- Transformation of snapshots onto a reference domain

$$\boldsymbol{\Phi}_{\boldsymbol{\mu}} : \Omega^r \to \Omega^{\boldsymbol{\mu}}, \boldsymbol{X}^r \mapsto \boldsymbol{X}^{\boldsymbol{\mu}} \qquad \boldsymbol{P}(\boldsymbol{X}^{\boldsymbol{\mu}}) = \boldsymbol{P}(\boldsymbol{\Phi}_{\boldsymbol{\mu}}(\boldsymbol{X}^r)) =: \boldsymbol{P}^r(\boldsymbol{X}^r)$$

- Effective stress can be obtained with

$$\begin{aligned} \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= |\Omega^{\boldsymbol{\mu}}|^{-1} \int_{\Omega^{\boldsymbol{\mu}}} \boldsymbol{P}(\boldsymbol{X}^{\boldsymbol{\mu}}; \bar{\boldsymbol{F}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) d\boldsymbol{X}^{\boldsymbol{\mu}} \\ &= |\Omega^{\boldsymbol{\mu}}|^{-1} \int_{\Omega^r} \boldsymbol{P}^r(\boldsymbol{X}^r; \bar{\boldsymbol{F}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \left|\det D\boldsymbol{\Phi}_{\boldsymbol{\mu}}\right| d\boldsymbol{X}^r \end{aligned}$$

Geometrical Parameters

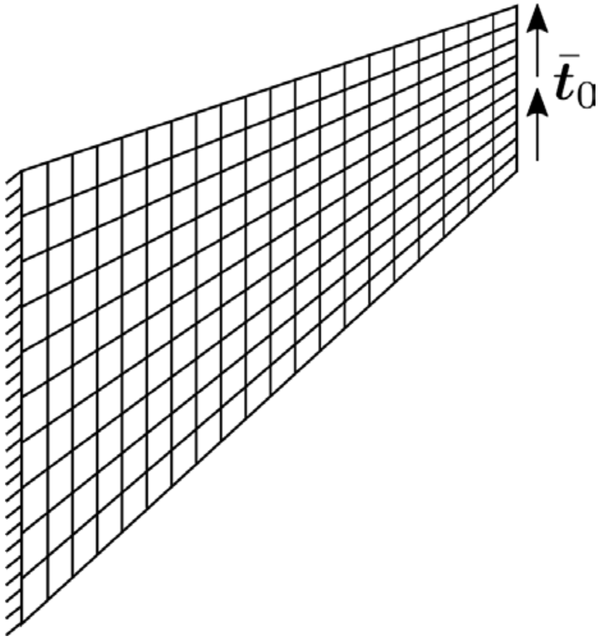


Problem Statement

Linear Momentum Balance $\bar{\boldsymbol{u}}(\bar{\boldsymbol{X}})$

$$
\begin{aligned}
\mathrm{Div}\,\bar{\boldsymbol{P}}(\bar{\boldsymbol{u}}) &= \mathbf{0} && \text{on } \bar{\Omega}, \\
\bar{\boldsymbol{P}}\bar{\boldsymbol{N}} &= \bar{\boldsymbol{t}}_0 && \text{on } \partial\bar{\Omega}^N, \\
\bar{\boldsymbol{u}} &= \bar{\boldsymbol{u}}_0 && \text{on } \partial\bar{\Omega}^D,
\end{aligned}
$$

$\bar{\boldsymbol{P}}$: Stress Tensor
$\bar{\boldsymbol{u}}$: Displacement
$\bar{\boldsymbol{F}}$: Deformation Gradient
$\boldsymbol{\mu}$: Parameters



Constitutive Model

$$
\bar{\boldsymbol{P}} = \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\mu})
$$

$$
\bar{\boldsymbol{F}} = \mathbf{I} + \frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{\boldsymbol{X}}}
$$

$$
\bar{\boldsymbol{A}} = \frac{\partial \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \bar{\boldsymbol{F}}}
$$

Highly **nonlinear** operator

Often not interpretable parameters $\boldsymbol{\mu}$

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]